Winter Notes on

MAT 217

Honours Linear Algebra

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Notes from my winter skim of *Linear Algebra Done Right* by Sheldon Axler, the reference textbook for Princeton University's MAT217: **Honours Linear Algebra** course.

Contents

1 Vector Spaces

1A $\mathbb{R}^n$ and $\mathbb{C}^n$

In this textbook, $\mathbb{R}$ and $\mathbb{C}$ are generalized to $\mathbb{F}$ as they are both **fields**, which is *any set* that contains at least 0 and 1 with operations of addition and multiplication satisfying the following properties:

- **commutativity**: $\alpha + \beta = \beta + \alpha \quad \forall \alpha, \beta \in \mathbb{F}$
- **associativity**: $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda) \quad \forall \alpha, \beta, \lambda \in \mathbb{F}$
- **identities**: $\lambda + 0 = \lambda$ and $1\lambda = \lambda \quad \forall \lambda \in \mathbb{F}$
- **additive inverse**: $\forall \alpha \in \mathbb{F}\ \exists!\ \beta \in \mathbb{F} : \alpha + \beta = 0$
- **multiplicative inverse**: $\forall \alpha \in \mathbb{F}\ \exists!\ \beta \in \mathbb{F} : \alpha\beta = 1$
- **distributive property**: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta \quad \forall \lambda, \alpha, \beta \in \mathbb{F}$

Extending our concept of fields to higher dimension, we define $\mathbb{F}^n$ as:

$$\mathbb{F}^n = \{(x_1, ..., x_n) : x_k \in \mathbb{F} \quad \forall k = 1, ..., n\}$$

1B Definition of Vector Spaces

(1.20) A vector space is any set V that is **closed** under addition and scalar multiplications. Additionally, it must also satisfy the following properties:

- **commutativity**: $u + v = v + u \quad \forall u, v \in V$
- **associativity**: $(u + v) + w = u + (v + w)$ and $(ab)u = a(bu) \quad \forall u, v, w \in V$ and $\forall a, b \in \mathbb{F}$
- **identities**: $v + 0 = v$ and $1v = v \quad \forall \lambda \in V$
- **additive inverse**: $\forall v \in V\ \exists\ w \in V : v + w = 0$
- **distributive property**: $a(u + u) = au + av \quad \forall u, v \in V$ and $\forall a \in \mathbb{F}$

Note the following:

- **(1.27)** We no longer demand the inverse to be unique as its **uniqueness** follows from *associativity*
- There is no multiplicative inverse defined for vector fields (obvious)

Other defintions:

- Denote $\mathbb{F}^S$ as the set of functions $f \colon S \to F$
- A vector space V is defined **over** a field (e.g. $\mathbb{F}$) as a place to draw its scalars

Other theorems:

- **(1.26)** A vector space has a unique additive identity which follows from *commutativity*
- **(1.30)** $0v = 0\ \forall v \in V$ which follows from the *existence of additive inverses*. Note that these two conditions are equivalent (replaceable) in the definition of a vector space (1B-5).
- **(1.31)** Similarly, $a0 = 0\ \forall a \in \mathbb{F}$, which again follows from *existence of additive inverses*
- **(1.32)** Lastly, $-1v = -v\ \forall v \in V$

1C Subspaces

In this book, V is assumed to be defined over $\mathbb{F}$, unless stated otherwise.

A subspace is the analogue of subset for vector spaces. A subspace U of V is defined as the *subset of* V which satisfies:

- **additive identity**: $0 \in U$
- **closed under addition**: $u, w \in U$ implies $u + w = U$
- **closed under multiplication**: $a \in \mathbb{F}$ and $u \in U$ implies $au \in U$

This reduced set of conditions is due to the underlying structure of V.

Subspace Sum

The analogue of set unions for vectorspaces are **subspace sums**, defined as:

$$\sum_{i-1}^{m} V_i = \left\{ \sum_{i=1}^{m} v_i : v_i \in V_i \quad i \in \{1, ..., m\} \right\}$$

or the set of all possible sums of elements of $\{V_i\}_1^m$.
Note that:

- **(1.40)** $\sum_{i-1}^{m} V_i$ is the **smallest subspace** of V containing V_1, ..., V_m
- **(1.45)** $V_1 \oplus ... \oplus V_m$ is **direct** $\iff$ the only way to write 0 as a sum $v_1 + ... + v_m$, where each $v_k \in V_k$, is by taking each v_k equal to 0
- **(1.46)** $U + W$ is a direct sum $\iff$ $U \cap W = \{0\}$. This relies *heavily* on existence of additive inverse and is only true for two subspaces

2 Finite-Dimensional Vector Spaces

2A Span and Linear Independence

Span

- **(2.2)** A *linear combination* of a list of vectors $v_k \in V$ is the vector of the form:

$$u = \sum_k a_k v_k$$

  where $a_k \in \mathbb{F}$.

- **(2.3)** The span of a list of vectors $v_k \in V$ is defined as:

$$\text{span}\left(\{v_k\}\right) = \left\{ \sum_k a_k v_k : a_k \in \mathbb{F} \right\}$$

- **(2.7)** If span($\{v_k\}$) $= V$, then $\{v_k\}$ *spans* V

- **(2.9)** A vector space is *finite-dimensional* if some list of vectors spans the space. By definition, lists have a finite length

Theorems:

- **(2.6)** $\text{span}(\{v_k\})$, $v_k \in V$ is the smallest subspace of V containing all v_k's
- Every subspace of a finite-dimensional vector space is finite-dimensional, which follows from **(2.19)** and **(2.22)**.

Polynomials

- **(2.10)** A function $p : \mathbb{F} \to \mathbb{F}$ is a polynomial with coefficients in $\mathbb{F}$ if $\exists \, \{a_k\}_0^m \in \mathbb{F}$ such that:

$$p(z) = \sum_{k=0}^{m} a_k z^k \quad \forall z \in \mathbb{F}$$

  The set $\mathcal{P}(\mathbb{F})$ is the set of all polynomials with coefficients in $\mathbb{F}$.

- **(2.11)** The degree of a polynomial is denoted by $\deg p$. The 0 polynomial has degree $\deg 0 = -\infty$.
- **(2.12)** $\mathcal{P}_m(\mathbb{F})$ denotes all polynomials with coefficients in $\mathbb{F}$ of degree at most m

Linear independence

(2.15) A list of vectors $\{v_k\}_1^m \in V$ is *linearly independent* if the only choice of $\{a_k\}_1^m \in \mathbb{F}$ that makes:

$$\sum_{k=1}^{m} a_k v_k = 0$$

is $a_1 = ... = a_m = 0$. The empty list ( ) is also **declared to be linearly independent**.

(2.17) A list is linearly **dependent** if it is not linearly independent.

(2.19) Linear dependence lemma

Notes:

- For a list of vectors $\{v_k\} \in V$, whether they are linearly independent depends also on the field $\mathbb{F}$ which V is defined over (2A-7).

2.19 Suppose $\{v_i\}_1^m \in V$ is linear dependent. Then there exists $k \in \{1, 2, ..., m\}$ such that:

$$v_k \in \text{span}(v_1, ..., v_{k-1})$$

Furthermore, if the k^{th} term is removed from $\{v_i\}_1^m$, then the span of the remaining list equals $\text{span}(\{v_i\}_1^m)$. This follows from the existence of a *non-zero* set of $\{a_i\}$ that makes the sum in **(2.15)** zero.

(2.22) len(linearly independent list) $\leq$ len(spanning list)

By far the most versatile theorem in this sub(chapter). This follows from the iterative nature of **(2.19)**.

2B Bases

(2.26) A *basis* of V is a list of vectors in V that is linearly independent **and** spans V.

- **(2.28)** A list of vectors $\{v_k\} \in V$ is a basis of V iff every $v \in V$ can be written uniquely in the form:
$$v = \sum_{k=1}^{n} a_k v_k$$
where $\{a_k\} \in \mathbb{F}$. Proof sketch:

  ($\Rightarrow$) linear independence guarantees uniqueness

  ($\Leftarrow$) uniqueness for $v = 0$ proves linear independence by **(2.15)**

- **(2.30)** Every spanning list can be reduced to a basis. Proof by iterative procedure using **(2.19)**.
- **(2.31)** Every finite-dimensional vector space has a basis, this follows from **(2.30)**
- **(2.32)** Every linearly independent list $\{u_k\} \in V$ extends to a basis. Proof outline:
  1. Append spanning list $\{w_i\} \in V$ and use **(2.30)**
  2. None of the u's gets removed because they are linearly independent (**2.19**)
- **(2.33)** For every subspace U of V, $\exists W$ such that $V = U \oplus W$.

2C Dimension

- **(2.34)** Any two basis of a finite-dimensional vector space have the same length, which follows from **(2.22)**.
- **(2.35)** The dimension of a vector space V, $\dim V$, is defined as the length of its basis

Suppose V is finite-dimensional and U is a subspace of V for the following theorems:

- **(2.37)** $\dim U \leq \dim V$. This follows from **(2.22)**
- **(2.38)** Every linearly independent list of vectors in V of length $\dim V$ is a basis of V.
- **(2.39)** If $\dim U = \dim V$, then $U = V$. This follows from **(2.38)**
- **(2.42)** Perhaps **less trivially**, every spanning list $\{v_k\}_1^n$ in V of length $\dim V$ is a basis of V. Proof outline
  1. Since $\{v_k\}_1^n$ is spanning, it can be reduced to basis **(2.30)**
  2. But every basis must have length n **(2.35)**, thus no elements are deleted.

(2.43) Dimension of a sum

If V_1 and V_2 are subspaces of a finite-dimensional vector space, then:
$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$
This is, perhaps, the most important theorem in this chapter. Proof for this is quite involved, see book (Axler, p. 47).

Note that $(V_1 + V_2) \cap V_3 \neq V_1 \cup V_3 + V_2 \cup V_3$, no matter how tempting it might be to assume so (2C-19).

3 Selected Problems

I will present my solution to the following selected problems: **1B-5, 2A-20, 2B-8, 2C-10, 2C-14, 2C-20**. Since I will be taking this course next semester (Spring 2025), I have not checked these solutions with any external source in accordance with the university's Honor Code. As such, these solutions might be erroneous.

1B-5 Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that:

$$0v = 0 \quad \forall v \in V \tag{1.30}$$

Solution To show that this condition is equivalent to the additive inverse condition, we will use (1.30) along with the other conditions to derive the existence of an additive inverse. *The other direction is not needed as (1.30) is a theorem of the definition (1.20).*
For a $v \in V$, we can construct $w = -1v \in V$ (closure of scalar multiplication). We have:

$$\begin{aligned} v + w &= 1v + -1v && \text{(multiplicative identity)} \\ &= (1-1)v && \text{(distributive)} \\ &= 0v && \text{(additive inverse for fields)} \\ &= 0 && (1.30) \end{aligned}$$

$\therefore \forall v \in V \ \exists w = -1v \in V : v + w = 0.$ □

2A-20 Suppose $p_0, p_1, ..., p_m$ are polynomials in $\mathcal{P}_m(\mathbb{F})$ such that $p_k(2) = 0$ for each $k \in \{0, ..., m\}$. Prove that $p_0, p_1, ..., p_m$ is **not linearly independent** in $\mathcal{P}_m(\mathbb{F})$.

Solution Since every $p_k(2) = 0 \ \forall k \in \{1, ..., m\}$, we can factorize $p_k = (x-2)f_k \ \forall k \in \{1, ..., m\}$, where f_k is some polynomial of degree $\deg f_k = \deg p_k - 1$. Thus, we can be sure that $f_k \in \mathcal{P}_{m-1}(\mathbb{F})$.

Suppose:

$$0 = \sum_{k=0}^{m} a_k p_k = (x-2)\sum_{k=0}^{m} a_k f_k \tag{Λ}$$

for some $\{a_k\} \in \mathbb{F}$. Since $\text{len}(f_0, ..., f_m) = m+1$, but $\text{len}(1, x, ..., x^{m-1}) = m$ is a spanning list of $\mathcal{P}_{m-1}(\mathbb{F})$. Thus $\{f_k\}_0^m$ cannot be linearly independent in $\mathcal{P}_{m-1}(\mathbb{F})$ by **(2.22)**.

Since (Λ) must be true $\forall x \in \mathbb{F}$, we can divide $(x-2)$ and get:

$$\sum_{k=0}^{m} a_k f_k = 0$$

As we have shown that $\{f_k\}_0^m$ is not linearly independent in $\mathcal{P}_{m-1}(\mathbb{F})$, $\exists \ \{a_k\}_0^m$ satisfying the above, where not all a_k's are zero.

Therefore, we have found a set of coefficients $\{a_k\}_0^m$ such that:

$$\sum_{k=0}^{m} a_k p_k = 0$$

Thus, $\{p_k\}_0^m$ are not linearly independent by **(2.17)**. □

2C-10 Suppose $m \in \mathbb{Z}^+$. For $0 \leq k \leq m$, let:

$$p_k(x) = x^k(1-x)^{m-k}$$

Show that $p_0, ..., p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.

*These are **Bernstein polynomials**, used to approximate continuous functions on* $[0,1]$

Solution Since the list has length $\dim \mathcal{P}_m(\mathbb{F})$, showing linear independence is sufficient by **(2.38)**. Consider $0 \in \mathcal{P}_m(\mathbb{F})$, we wish to show:

$$0 = a_0(1-x)^m + ... + a_k x^k(1-x)^{m-k} + ... + a_m x^m$$

only for $a_0 = ... = a_m = 0$ **(2.15)**. Notice that this expands to:

$$0 = a_0 + (c_{10}a_0 + c_{11}a_1)x + ... + \left(\sum_{j=0}^{k} c_{kj}a_j\right)x^k + ... + \left(\sum_{j=0}^{m} c_{kj}a_j\right)x^m$$

Since we know that $(1, x, ..., x^m)$ is a basis of $\mathcal{P}_m(\mathbb{F})$, we must have:

$$\sum_{j=0}^{k} c_{kj}a_j = 0 \qquad \forall k \in \{0, ..., m\}$$

At $k = 0$, the sum evaluates to $a_0 = 0$.
For a generic k, where $a_0 = ... = a_k = 0$, we have:

$$\begin{aligned}\sum_{j=0}^{(k+1)} c_{(k+1)j}a_j &= \sum_{j=0}^{k} c_{(k+1)j}a_j + c_{(k+1)(k+1)}a_{k+1} \\ &= 0 + c_{(k+1)(k+1)}a_{k+1} = 0\end{aligned}$$

$$\boxed{\Rightarrow a_{k+1} = 0}$$

$\therefore$ By induction, $a_0 = ... = a_m = 0$ □

2C-14 Suppose V is a ten-dimensional vector space and V_1, V_2, V_3 are subspaces of V with $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

Solution Since $V_i + V_j$ is a subspace of V **(1,40)**, we have:

$$\begin{aligned} \dim(V_i + V_j) &\leq \dim V = 10 && (2.37)\\ \dim V_i + \dim V_j - \dim(V_i \cap V_j) &\leq 10 && (2.43)\\ 14 - \dim(V_i \cap V_j) &\leq 10 && \text{(given)}\\ 4 &\leq \dim(V_i \cap V_j) && (\Delta) \end{aligned}$$

We wish to show $\dim(V_1 \cap V_2 \cap V_3) > 0$. Consider the following sum $V_1 \cap V_2 + V_3$, which is a subspace of V_3:

$$\begin{aligned} \dim(V_1 \cap V_2 + V_3) &\leq \dim V = 10 && (2.37)\\ \dim(V_1 \cap V_2) + \dim V_3 - \dim(V_1 \cap V_2 \cap V_3) &\leq 10 && (2.43)\\ 4 + 7 - \dim(V_1 \cap V_2 \cap V_3) &\leq 10 && (\Delta)\\ 1 &\leq \dim(V_1 \cap V_2 \cap V_3) \end{aligned}$$

Thus, $\dim(V_1 \cap V_2 \cap V_3) \geq 1 > 0$. □

2C-20 Prove that if V_1, V_2, and V_3 are subspaces of a finite-dimensional vector space, then:

$$\dim\left(\sum_{i=1}^{3} V_i\right) = \sum_{i=1}^{3} \dim V_i - \frac{1}{3} \sum_{\substack{i,j \in 1,2,3 \\ i<j}} \dim(V_i \cap V_j) - \frac{1}{3} \sum_{\substack{i,j,k \in \{1,2,3\} \\ i \neq j \neq k}} \dim\left((V_i \cap V_j) \cap V_k\right)$$

Solution Consider the following subspace sum:

$$\begin{aligned} \dim\left((V_1 + V_2) + V_3\right) &= \dim(V_1 + V_2) + \dim V_3 - \dim\left((V_1 + V_2) \cap V_3\right)\\ &= \sum_{i=1}^{3} \dim V_i - \dim(V_1 \cap V_2) - \dim\left((V_1 + V_2) \cap V_3\right) \end{aligned}$$

Varying the order of summing in the left-hand side, we obtain two slightly different right-hand sides. Adding these together and dividing throughout by 3 gives us the desired expression. □